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Letter to the Editor
Comments on "Nonlocal strain softening bar revisited" by Christer Nilsson [International Journal of Solids and Structures 34 (1997) 4399–4419]

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Nilsson (1997) addressed the analytical solution of an elastic plastic softening bar in extension within the framework of nonlocal plasticity, making use to this purpose of a theory of nonlocal plasticity preliminary devised in the same paper (Section 2). The results there presented—from the thermodynamic arguments supporting the devised theory, to the bar solution—are quite interesting. The writers, involved in the same research area (Polizzotto et al., 1997; Polizzotto and Borino, 1998), wish to discuss a few points of the paper in a fully collaborative spirit. We use the same notation as in Nilsson's paper.

1. The nonlocal plastic strain

Nilsson refers to a restricted nonlocality in which total and elastic strains are local, whereas plastic strain and a scalar internal variable (accumulated plastic strain) are not. The writers believe that there are not enough motivations in order to treat plastic strain as nonlocal within a theory of nonlocal plasticity envisaged as a localization limiter. The decomposition of the local total strain into a local elastic part and a nonlocal plastic part seems not to be a consistent operation. Bažant and Pijaudier-Cabot (1988) and Bažant and Lin (1988) state that a nonlocal plasticity theory, in which the accumulated plastic strain is the only nonlocal variable, is adequate. There is no need to make things more complex than necessary. (But in the following we accept the Author's idea to treat plastic strain as nonlocal.)

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2. The dissipation function

Nilsson remarks, in the first paragraph of Section 2, the necessity to account for nonlocality residuals in order to derive correct local equations; however he states eqns (26) and (27) without introducing any such residual, through an argument aimed at just one to get rid of it. In the writers' view, when we pass from $(24)_1$ to a local expression as (26) we have to introduce the invoked nonlocality residual and write, instead of (26),

$$D^p = \overline{\{\boldsymbol{\sigma}\}}_p \cdot \dot{\boldsymbol{\varepsilon}}^p + \overline{\{q\}}_h \dot{\kappa} + P \geqslant 0 \quad \text{in } B \tag{93}$$

where P denotes the nonlocality residual satisfying

$$\int_B P \, \mathrm{d}V = 0. \tag{94}$$

It is the role of the nonlocality residual to guarantee that the Clausius–Duhem inequality (10) be satisfied locally, i.e. $-\psi + \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\varepsilon}} + P \geqslant 0$, everywhere in B. The latter inequality may be utilized to arrive at (93) without passing through $(24)_1$. On the other hand considering that the plastic deformation mechanism is locally described by $\dot{\boldsymbol{\varepsilon}}^p$ and $\dot{\kappa}$ which are the independent evolutive variables (i.e. to be governed by the plastic flow laws), the plastic dissipation density D^p must take on a bilinear form as

$$D^p = \mathbf{S} \cdot \dot{\boldsymbol{\varepsilon}}^p + Q\dot{\kappa}, \quad \text{in } B \tag{95}$$

where $\mathbf{S}$ and Q are some (nonlocal) thermodynamic forces to be identified. Comparing (93) and (95) with each other gives

$$P = \mathbf{S} \cdot \dot{\boldsymbol{\varepsilon}}^p + Q\dot{\kappa} - \overline{\{\boldsymbol{\sigma}\}}_p \cdot \dot{\boldsymbol{\varepsilon}}^p - \overline{\{q\}}_h \dot{\kappa} \tag{96}$$

and substituting this expression into (94), we have

$$\int_B [(\mathbf{S} - \overline{\{\boldsymbol{\sigma}\}}_p) \cdot \dot{\boldsymbol{\varepsilon}}^p + (Q - \overline{\{q\}}_h)\dot{\kappa}] \, \mathrm{d}V = 0. \tag{97}$$

Since (97) must hold for any choice of the plastic mechanism $(\dot{\boldsymbol{\varepsilon}}^p, \dot{\kappa})$, we obtain

$$\mathbf{S} = \overline{\{\boldsymbol{\sigma}\}}_p, \quad Q = \overline{\{q\}}_h \quad \text{in } B. \tag{98}$$

As a consequence, it is $P \equiv 0$, and D^p of either (93) and (95) coincides with (26) and complies with (27). The consequence of the above is that—according to the writers' view—the zero-residual expression of D^p given by (26) and (27) is exact for the proposed model. Note that, had we used $(24)_2$ instead of $(24)_1$ and written (93) as

$$D^p = \boldsymbol{\sigma} \cdot \langle \dot{\boldsymbol{\varepsilon}}^p \rangle + q \langle \dot{\kappa} \rangle + P \tag{99}$$

we would have obtained the same result as (98) but

$$P = \overline{\{\boldsymbol{\sigma}\}}_p \cdot \dot{\boldsymbol{\varepsilon}}^p + \overline{\{q\}}_h \dot{\kappa} - \boldsymbol{\sigma} \cdot \langle \dot{\boldsymbol{\varepsilon}}^p \rangle - q \langle \dot{\kappa} \rangle. \tag{100}$$

3. The plastic flow laws

These laws are aimed to provide the evolutive variables $\dot{\boldsymbol{\varepsilon}}^p$ and $\dot{\kappa}$, at any given state of the material. In the writers' opinion, a consistent way to write the plastic flow laws for nonlocal plasticity is to relate the variables $(\dot{\boldsymbol{\varepsilon}}^p, \dot{\kappa})$ to the corresponding thermodynamic forces, $(\overline{\{\boldsymbol{\sigma}\}}_p, \overline{\{q\}}_h)$, displayed by the zero-residual bilinear-form dissipation, D^p, in (26), through a convex yield function having as arguments these same forces, i.e. $f(\overline{\{\boldsymbol{\sigma}\}}_p, \overline{\{q\}}_h) \leqslant 0$. The simple application of the normality rule then gives:

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda} \frac{\partial f}{\partial \overline{\{\boldsymbol{\sigma}\}}_p}, \quad \dot{\kappa} = \dot{\lambda} \frac{\partial f}{\partial \overline{\{q\}}_h} \quad \text{in } B \tag{101a}$$

$$f(\overline{\{\boldsymbol{\sigma}\}}_p, \overline{\{q\}}_h) \leqslant 0, \quad \dot{\lambda} \geqslant 0, \quad \dot{\lambda} f(\overline{\{\boldsymbol{\sigma}\}}_p, \overline{\{q\}}_h) = 0 \quad \text{in } B, \tag{101b}$$

which describe the constitutive behaviour of the entire set of material particles included in B. The nonlocal nature of (101) is twofold: first because (101) are a set of domain equations, second because the material state is there described by the (nonlocal) weighted stresses.

Author, in Subsection 2.2.2, proposed a different procedure to derive the plastic flow rules (32). To this purpose, he adopts a yield function as $f(\boldsymbol{\sigma}, q) \leqslant 0$ with driving forces $(\boldsymbol{\sigma}, q)$ and tries to derive these plastic flow rules as the Kuhn–Tucker conditions of a maximum dissipation principle as in (29).

These writers wish to demonstrate that choosing the local stresses $\boldsymbol{\sigma}$, q as driving forces and a yield function as $f(\boldsymbol{\sigma}, q) \leqslant 0$ leads not to (32) but rather to plastic flow rules for the nonlocal evolutive variables $\langle \dot{\boldsymbol{\varepsilon}}^p \rangle$, $\langle \dot{\kappa} \rangle$, as it can be expected from $(24)_2$ where $\boldsymbol{\sigma}$, q appear as thermodynamic forces corresponding to $\langle \dot{\boldsymbol{\varepsilon}}^p \rangle$, $\langle \dot{\kappa} \rangle$, respectively.

Following the Lagrange multiplier method as in Nilsson's paper (but with more precise details), we address the problem:

$$\max_{(\sigma, q)} D^p(\boldsymbol{\sigma}, q; \dot{\boldsymbol{\varepsilon}}^p, \dot{\kappa}) \quad \text{s.t. } f(\boldsymbol{\sigma}, q) \leqslant 0 \quad \text{in } B \tag{102}$$

where D^p is the functional

$$D^p = \left(\int_B \tilde{w}^p(\mathbf{x}, \mathbf{z}) \boldsymbol{\sigma}(\mathbf{z}) \, \mathrm{d}V(\mathbf{z}) \right) \cdot \dot{\boldsymbol{\varepsilon}}^p(\mathbf{x}) + \left(\int_B \tilde{w}^h(\mathbf{x}, \mathbf{z}) q(\mathbf{z}) \, \mathrm{d}V(\mathbf{z}) \right) \dot{\kappa}(\mathbf{x}) \tag{103}$$

$\mathbf{x}$ being any fixed point in B. The Lagrangian reads

$$L^p = -D^p(\boldsymbol{\sigma}, q; \dot{\boldsymbol{\varepsilon}}^p, \dot{\kappa}) + \frac{1}{V} \int_B \dot{\gamma}(\mathbf{x}, \mathbf{z}) f(\boldsymbol{\sigma}(\mathbf{z}), q(\mathbf{z})) \, \mathrm{d}V(\mathbf{z}) \tag{104}$$

where $\dot{\gamma}(\mathbf{x}, \mathbf{z}) \geqslant 0$ is the pertinent Lagrangian multiplier and the (irrelevant) division by $V = V(B)$ is due to dimensionality reasons. The first variation of L^p is

$$\delta L^p = \int_B \delta\boldsymbol{\sigma}(\mathbf{z}) \cdot \left[-\tilde{w}^p(\mathbf{x},\mathbf{z})\dot{\boldsymbol{\varepsilon}}^p(\mathbf{x}) + \frac{1}{V}\dot{\gamma}(\mathbf{x},\mathbf{z}) \left.\frac{\partial f}{\partial \boldsymbol{\sigma}}\right|_{\mathbf{z}} \right] \mathrm{d}V(\mathbf{z})$$

$$+ \int_B \delta q(\mathbf{z}) \left[-\tilde{w}^h(\mathbf{x},\mathbf{z})\dot{\kappa}(\mathbf{x}) + \frac{1}{V}\dot{\gamma}(\mathbf{x},\mathbf{z}) \left.\frac{\partial f}{\partial q}\right|_{\mathbf{z}} \right] \mathrm{d}V(\mathbf{z}) + \int_B \delta\dot{\gamma}(\mathbf{x},\mathbf{z}) f(\boldsymbol{\sigma}(\mathbf{z}), q(\mathbf{z})) \, \mathrm{d}V(\mathbf{z}). \quad (105)$$

Thus, the optimality conditions are:

$$\tilde{w}^p(\mathbf{x},\mathbf{z})\dot{\boldsymbol{\varepsilon}}^p(\mathbf{x}) = \frac{1}{V}\dot{\gamma}(\mathbf{x},\mathbf{z}) \left.\frac{\partial f}{\partial \boldsymbol{\sigma}}\right|_{\mathbf{z}}$$

$$\tilde{w}^h(\mathbf{x},\mathbf{z})\dot{\kappa}(\mathbf{x}) = \frac{1}{V}\dot{\gamma}(\mathbf{x},\mathbf{z}) \left.\frac{\partial f}{\partial q}\right|_{\mathbf{z}} \quad (106a)$$

$$f(\boldsymbol{\sigma}, q)|_{\mathbf{z}} \leqslant 0, \quad \dot{\gamma}(\mathbf{x},\mathbf{z}) \geqslant 0, \quad \dot{\gamma}(\mathbf{x},\mathbf{z}) f(\boldsymbol{\sigma}, q)|_{\mathbf{z}} = 0 \quad (106b)$$

which hold for all $\mathbf{z} \in B$ and every $\mathbf{x} \in B$. The apparently strange form of (106a) is due to the particular format given to the maximum dissipation principle, indeed not consistent with the dissipation representation $(24)_2$. But we can equally derive the pertinent result by integration of (106) with respect to $\mathbf{x} \in B$. Remembering (8) and with the notation

$$\dot{\lambda}(\mathbf{z}) = \frac{1}{V}\int_B \dot{\gamma}(\mathbf{x},\mathbf{z}) \, \mathrm{d}V(\mathbf{x}), \quad (107)$$

we obtain, for all points $\mathbf{z} \in B$,

$$\langle \dot{\boldsymbol{\varepsilon}}^p \rangle = \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\sigma}} \quad \langle \dot{\kappa} \rangle = \dot{\lambda}\frac{\partial f}{\partial q} \quad (108a)$$

$$f(\boldsymbol{\sigma}, q) \leqslant 0, \quad \dot{\lambda} \geqslant 0 \quad \dot{\lambda} f(\boldsymbol{\sigma}, q) = 0. \quad (108b)$$

Equations (32) and (34) are somewhat different than (108a). The procedure leading to (32) in the paper should perhaps be reconsidered.

4. The maximum dissipation principle

Once the appropriate driving forces for plastic yielding have been specified (i.e. $\overline{\{\sigma\}}_p$ and $\overline{\{q\}}_h$ in our approach), the format to give to the above principle is, in the writers' opinion, strictly related to the way in which the assigned plastic mechanism is described, as well as to the format of the bilinear form taken on by the corresponding dissipation. For instance, if we assign the mechanism through the local variables $\dot{\boldsymbol{\varepsilon}}^p$, $\dot{\kappa}$, we have to use D^p of $(24)_1$ and write:

$$\mathbb{D}^p[\dot{\boldsymbol{\varepsilon}}^p, \dot{\kappa}] = \max_{(\sigma,q)} \int_B (\overline{\{\boldsymbol{\sigma}\}}_p \cdot \dot{\boldsymbol{\varepsilon}}^p + \overline{\{q\}}_h \dot{\kappa}) \, \mathrm{d}V \quad s.t. \, f(\overline{\{\boldsymbol{\sigma}\}}_p, \overline{\{q\}}_h) \leqslant 0 \quad \text{in } B \quad (109)$$

which admits as Kuhn–Tucker conditions the equation set (101), the latter being the corresponding

plastic flow rules. But, if the plastic mechanism is assigned through nonlocal strain variables, say $\dot{\boldsymbol{\varepsilon}}^p_{NL}$ and $\dot{\kappa}_{NL}$, we have the use $(24)_2$ and the principle takes on the form:

$$\mathbb{D}^p[\dot{\boldsymbol{\varepsilon}}^p_{NL}, \dot{\kappa}_{NL}] \cdot \max_{(\sigma,q)} \int_B (\boldsymbol{\sigma} \cdot \dot{\boldsymbol{\varepsilon}}^p_{NL} + q\dot{\kappa}_{NL})\, \mathrm{d}V \quad s.t.\, f(\overline{\{\boldsymbol{\sigma}\}}_p, \overline{\{q\}}_h) \leqslant 0 \quad \text{in } B \tag{110}$$

which can be shown to admit the Kuhn–Tucker conditions:

$$\dot{\boldsymbol{\varepsilon}}^p_{NL} = \left\langle \dot{\lambda} \frac{\partial f}{\partial \overline{\{\boldsymbol{\sigma}\}}_p} \right\rangle, \quad \dot{\kappa}_{NL} = \left\langle \dot{\lambda} \frac{\partial f}{\partial \overline{\{q\}}_h} \right\rangle \quad \text{in } B \tag{111a}$$

$$f(\overline{\{\boldsymbol{\sigma}\}}_p, \overline{\{q\}}_h) \leqslant 0, \quad \dot{\lambda} \geqslant 0, \quad \dot{\lambda} f(\overline{\{\boldsymbol{\sigma}\}}_p, \overline{\{q\}}_h) = 0 \quad \text{in } B \tag{111b}$$

where the symbol $\langle \cdot \rangle$ denotes the operation in (8), i.e. it is

$$\dot{\boldsymbol{\varepsilon}}^p_{NL} = \langle \dot{\boldsymbol{\varepsilon}}^p \rangle, \quad \dot{\kappa}_{NL} \langle \dot{\kappa} \rangle \quad \text{in } B \tag{112}$$

with $\dot{\boldsymbol{\varepsilon}}^p$, $\dot{\kappa}$ given by (101a).

The inconsistency of the maximum dissipation principle as expressed by (29), with the ensuing procedure to arrive at (32), is evidentiated by the fact that the plastic work obtained from (32) does not coincide with (26), as in fact

$$\beta^p \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\varepsilon}}^p + \beta^h q \dot{\kappa} \neq \overline{\{\boldsymbol{\sigma}\}}_p \cdot \dot{\boldsymbol{\varepsilon}}^p + \overline{\{q\}}_h \dot{\kappa}. \tag{113}$$

5. The bar solution

In Nilsson's paper, Section 3, the central cross section of the uniform bar is given a reduced resistance such as to force the local plastic strain to be activated only in this section. The nonlocal plastic strain is on the contrary distributed throughout the bar length with a rapidly decaying pattern from the middle cross section source to the bar ends, but in effect remaining different from zero everywhere. Such a constitutive behaviour, by which the locally produced plastic strain is redistributed in the whole structure's volume, or in large portions of it around the source points, makes the proposed nonlocal plasticity model to rule out strain localization phenomena, rather than to limit them. The criterion used in the paper to find the 'width of the localized zone' confirms that such a zone does not exist as a strict material feature. In fact, this criterion is equivalent to stating: the width of the strain localization band is that of the bar segment where—in the softening regime—the total strain does not decrease, i.e. $\dot{\varepsilon} \geqslant 0$, with increasing kinematic load Δu, and thus where the (nonlocal) plastic strain, $\langle \varepsilon^p \rangle$, increases not less than the elastic strain σ/E decreases. Such a criterion will likely lead to a localization bandwidth which in general depends on the structural problem features more than size effect considerations may justify; additionally, it apparently is not unique.

Solutions of the above bar problem in the softening regime have been given in the literature (e.g. Bažant and Lin, 1988; de Borst and Mühlhaus, 1992; Polizzotto and Borino, 1998) and all of them are characterized by a softening plastic band the length of which is a strict material parameter, whereas the whole remaining part of the bar unloads elastically. The solution presented

in the paper has no such characteristics. These writers wonder whether the introduction of nonlocality for plastic strain is responsible for such a discrepancy.

6. Conclusions

Thermodynamic aspects of nonlocal (and gradient) plasticity, and the consistency of the related plastic flow laws as well, are main points under debate at present. The writer's view on these points can be, in conclusion, be summarized as follows with reference to nonlocal plasticity (see Polizzotto and Borino, 1998; Polizzotto et al., 1997a,b):

(1) For a nonlocal plasticity theory as a strain localization limiter, it is sufficient to treat as nonlocal the scalar internal variable attached to the isotropic hardening, see Bažant and Pijaudier-Cabot (1988) and Bažant and Lin (1988).
(2) The Clausius–Duhem inequality holds globally for any subdomain of the body en-compassing the region(s) of it where a diffuse plastic deformation mechanism takes place and which cannot become smaller than some limit related to the material internal length scale. However, the above inequality can be given a point-wise format provided a suitable nonlocality residual is introduced. The usual thermodynamic arguments can then be applied to derive the relevant state equations and dissipation density.
(3) This dissipation density includes the nonlocality residual as an additional term. The latter can be determined observing that—within the (insulated) region where a diffuse plastic mechanism takes place—the dissipation density can be expressed as a bilinear form in terms of the local evolutive variables and the related thermodynamic forces.
(4) The latter thermodynamics forces are the appropriate driving forces for plastic strain and thus the arguments of the yield function (and of the plastic potential function in case of non-associated plasticity).
(5) The plastic flow laws can be written straightforwardly by application of the normality rule (or in analogy with it in case of nonassociated plasticity). At difference with local plasticity (in which these laws concern the constitutive behaviour of a single material particle), in nonlocal plasticity the plasticity flow laws are domain equations governing the behaviour of the entire collection of material particles.
(6) For associated nonlocal plasticity, a maximum dissipation principle can be written considering the overall dissipation of the body. Its specific format depends on the way the assigned diffuse plastic mechanism is specified in the body either with local, or nonlocal, evolutive variables.

The theory here above synthesized is centred upon the initial choice of the local evolutive variables (e.g. $\dot{\boldsymbol{\varepsilon}}^p$, $\dot{\kappa}$) and of the (regularization) operator(s) transforming these local variables into nonlocal ones (e.g. $\dot{\boldsymbol{\varepsilon}}^p \Rightarrow \langle \dot{\boldsymbol{\varepsilon}}^p \rangle$, $\dot{\kappa} \Rightarrow \langle \dot{\kappa} \rangle$ in Nilsson's approach). It turns out to possess a unified character since the mentioned regularization operator(s) can be chosen, within certain limits, arbitrarily, either as an integral operator for nonlocal plasticity or as a differential operator for gradient plasticity. It is worth mentioning that, in case of gradient plasticity, the procedure envisaged at point (3) above to determine the nonlocality residual leads also to the (nonambiguous) thermodynamic boundary conditions which must be appended to the plastic flow laws at point (5); furthermore, the maximum dissipation principle of point (6) contains suitable boundary

conditions as additional constraints. This enhanced potentiality of the writers' theory can be envisaged as a manifestation of its thermodynamic consistency.

The results presented by Nilsson (1997) provide an effective stimulus within a rather new research area, now in rapid expansion. The present discussion indicates that some of the hypotheses on which Nilsson's paper is based seem not to be in full agreement with thermodynamics.

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